An Open Problem on Metric Invariants of Tetrahedra

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ABSTRACT

In ISSAC 2000, P. Lisoněk and R.B. Israel [3] asked whether, for *any* given positive real constants V, R, A_1, A_2, A_3, A_4, there are always *finitely many* tetrahedra, all having these values as their respective volume, circumradius and four face areas. In this paper we present a negative solution to this problem by constructing a family of tetrahedra $T_{(x,y)}$ where (x, y) varies over a component of a cubic curve such that all tetrahedra $T_{(x,y)}$ share the same volume, circumradius and face areas.

Categories and Subject Descriptors: G.0 [General]

General Terms: Algorithms

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1. INTRODUCTION

Consider a tetrahedron $T = P_1P_2P_3P_4$ in $\mathbf{R}^3$. Let $d_{i,j} = d(P_i, P_j)$ be the distance between vertices P_i and P_j. It is well known that the volume V of tetrahedron T can be expressed by the *Cayley-Menger determinant* associated with the points $P_l (1 \le l \le 4)$ as follows:

$$V^2 = \frac{1}{288}\begin{vmatrix} 0 & d_{1,2}^2 & d_{1,3}^2 & d_{1,4}^2 & 1 \\ d_{2,1}^2 & 0 & d_{2,3}^2 & d_{2,4}^2 & 1 \\ d_{3,1}^2 & d_{3,2}^2 & 0 & d_{3,4}^2 & 1 \\ d_{4,1}^2 & d_{4,2}^2 & d_{4,3}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

For convenience, let M_0 denote the above determinant and M_l the principal minor determinant of M_0 obtained by deleting the l-th row and l-th column of M_0 for $l = 1, \cdots, 5$. Then, the circumradius R, i.e., the radius of the sphere circumscribed to T, can be expressed by $d_{i,j}$ in the following form:

$$R^2 = -\frac{M_5}{2M_0}.$$

This means that the square of the volume of a tetrahedron is a polynomial in $d_{i,j}^2$ with rational coefficients, and the square of the circumradius is a rational function in $d_{i,j}^2$. According to Heron formula of triangles, the squares of the areas of four faces of the tetrahedron are also rational polynomials in $d_{i,j}^2$:

$$A_l^2 = -\frac{1}{16}M_l, \quad l = 1, 2, 3, 4.$$

Call $d_{i,j}, V, R, A_l$ the metric invariants of a tetrahedron. Since M_l can be regarded as polynomials in $d_{i,j}$, we have 6 polynomials connecting these metric invariants:

$$\begin{aligned} f_1(d_{1,2}, d_{1,3}, \cdots, d_{3,4}, V) &= 288V^2 - M_0, \\ f_2(d_{1,2}, d_{1,3}, \cdots, d_{3,4}, R) &= 2M_0R^2 + M_5, \\ f_3(d_{1,2}, d_{1,3}, \cdots, d_{3,4}, A_1) &= 16A_1^2 + M_1, \\ f_4(d_{1,2}, d_{1,3}, \cdots, d_{3,4}, A_2) &= 16A_2^2 + M_2, \\ f_5(d_{1,2}, d_{1,3}, \cdots, d_{3,4}, A_3) &= 16A_3^2 + M_3, \\ f_6(d_{1,2}, d_{1,3}, \cdots, d_{3,4}, A_4) &= 16A_4^2 + M_4. \end{aligned}$$

According to a theorem in [1] on embedding a simplex in $\mathbf{R}^n$, for any given positive real constants V, R, A_1, A_2, A_3, A_4, there exists a tetrahedron $T = P_1P_2P_3P_4$ in $\mathbf{R}^3$ such that $P_iP_j = d_{i,j}$, whenever there is a solution $(d_{1,2}, d_{1,3}, \cdots, d_{3,4})$ with $d_{i,j} > 0$ to the system $\{f_1 = 0, f_2 = 0, \cdots, f_6 = 0\}$.

In [2], a question proposed by M. Mazur asked whether or not a tetrahedron is uniquely determined by its volume, circumradius and face areas. P. Lisoněk and R.B. Israel [3] gave a negative answer to this question by constructing two or more tetrahedra that share the same volume, circumradius and face areas, and suggested to discuss whether, for *any* positive real constants V, R, A_1, A_2, A_3, A_4, there are *finitely many* tetrahedra, all having these values as their respective metric invariants.

In this paper, we present a negative solution to Lisoněk and Israel's problem.

2. A MANIFOLD SOLUTION TO THE METRIC INVARIANT EQUATIONS

Our main result is the following theorem.

THEOREM 1. *Let*

$$G(x, y) = 3\,(1 - x)(17 - 18y)(1 + 3x + 3y) - 9x^2 - 3x - 37,$$

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$$g_{1,2} = 324\,(1-y)\,(1+y),$$
$$g_{1,3} = 324\,(1-x)\,(1+x),$$
$$g_{1,4} = (29-18x-18y)\,(7+18x+18y),$$
$$g_{2,3} = 36\,(7-3x-3y)\,(1+3x+3y),$$
$$g_{2,4} = (17-18x)\,(31+18x),$$
$$g_{3,4} = (17-18y)\,(31+18y),$$

and

$$G_0 = \{\,(x,y) \in \mathbf{R}^2 \mid G(x,y)=0,\ |x|<1,\ |y|<1\}.$$

Then, for each $(\xi,\eta) \in G_0$*, it holds that*

$$g_{1,2}(\xi,\eta) > 0, g_{1,3}(\xi,\eta) > 0, \cdots, g_{3,4}(\xi,\eta) > 0,$$

and there exists a tetrahedron $T_{(\xi,\eta)}$ *with edge-lengths* $\sqrt{g_{i,j}(\xi,\eta)}$ $(1 \le i < j \le 4)$*, whose volume, circumradius and four face areas are equal to*

$$441,\ \frac{43\sqrt{3}}{6},\ 84\sqrt{3},\ 63\sqrt{3},\ 63\sqrt{3},\ 63\sqrt{3},$$

respectively.

One can see that the polynomial $G(x,y)$ is symmetric in x and y by expanding it. The shape of the semi-algebraic set G_0 looks like a UFO, as shown in Fig. 1. Theorem 1 means that there are a family of infinitely many tetrahedra which share the same volume, circumradius and face areas. This presents a negative answer to Lisoněk and Israel's question. To prove the theorem, we need verify the following lemmas.

LEMMA 1. *It holds for all* $(\xi,\eta) \in G_0$ *that*

$$-\frac{1}{2} < \xi < \frac{3}{4},\ -\frac{1}{2} < \eta < \frac{3}{4},$$

where G_0 *is defined in Theorem 1.*

PROOF OF LEMMA 1: The set G_0 is not empty since

$$\left(\frac{1}{4}+\frac{\sqrt{2}}{3},\ \frac{1}{4}-\frac{\sqrt{2}}{3}\right) \in G_0.$$

The curve $G(x,y)=0$ does not intersect either of the lines $x=1, x=-1, y=1, y=-1$ because none of $G(1,y)$, $G(-1,y), G(x,1), G(x,-1)$ has a real zero. So G_0 is compact. For all $(\xi,\eta) \in G_0$, the maximum and minimum of ξ both are real zeros of the polynomial obtained by eliminating η from $G(\xi,\eta)$ and $\frac{\partial}{\partial\eta}G(\xi,\eta)$, that is,

$$324\,\xi^3 + 576\,\xi^2 - 275\,\xi - 233,$$

which has only 2 real zeros in $(-1,1)$ as follows,

$$-0.48681\cdots,\quad 0.73069\cdots.$$

So we have $-\frac{1}{2} < \xi < \frac{3}{4}$, analogously, $-\frac{1}{2} < \eta < \frac{3}{4}$.

LEMMA 2. *It holds for all* $(\xi,\eta) \in G_0$ *that*

$$g_{i,j}(\xi,\eta) > 0, \quad (1 \le i < j \le 4)$$

where G_0 *and* $g_{i,j}$ *are defined in Theorem 1.*

PROOF OF LEMMA 2: The first two inequalities

$$g_{1,2}(\xi,\eta) = 324\,(1-\eta)\,(1+\eta) > 0,$$
$$g_{1,3}(\xi,\eta) = 324\,(1-\xi)\,(1+\xi) > 0$$

are trivial since $|\xi|<1$ and $|\eta|<1$.

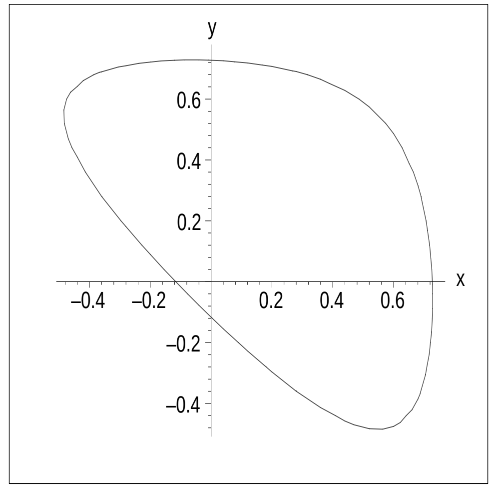

Fig. 1

The last two,

$$g_{2,4}(\xi,\eta) = (17-18\,\xi)\,(31+18\,\xi) > 0,$$
$$g_{3,4}(\xi,\eta) = (17-18\,\eta)\,(31+18\,\eta) > 0,$$

also hold because $|\xi| < \frac{3}{4}$, $|\eta| < \frac{3}{4}$, by Lemma 1. Next, observe that $-9x^2-3x-37$, the sum of the last three terms of $G(x,y)$, is always negative, whenever $G(x,y)=0$, the following inequality holds:

$$3\,(1-x)(17-18y)(1+3x+3y) > 0.$$

We have known $(1-\xi)(17-18\eta) > 0$ for $(\xi,\eta) \in G_0$, hence $1+3\xi+3\eta > 0$, and then

$$g_{2,3}(\xi,\eta) = 36\,(7-3\xi-3\eta)(1+3\xi+3\eta) > 0.$$

Furthermore, $1+3\xi+3\eta > 0$ implies $7+18\xi+18\eta > 0$ and that $|\xi| < \frac{3}{4}$ and $|\eta| < \frac{3}{4}$ imply $29-18\xi-18\eta > 0$, so we have

$$g_{1,4}(\xi,\eta) = (29-18\xi-18\eta)(7+18\xi+18\eta) > 0.$$

This completes the proof of Lemma 2.

LEMMA 3. *The assignment* $\{\,d_{i,j} = \sqrt{g_{i,j}(\xi,\eta)},\ (1 \le i < j \le 4),\ V=441,\ R=\frac{43\sqrt{3}}{6}, A_1 = 84\sqrt{3},\ A_2 = 63\sqrt{3},\ A_3 = 63\sqrt{3},\ A_4 = 63\sqrt{3}\}$ *solves the system* $\{\,f_1,f_2,f_3,f_4,f_5,f_6\}$ *for every* $(\xi,\eta) \in G_0$*.*

This is simple, just recall $G(\xi,\eta)=0$ on doing substitution.

PROOF OF THEOREM 1: Denote the Cartesian coordinates to be determined of the vertices P_1, P_2, P_3, P_4 by

$$(\,0,0,0\,),\ (x_1,0,0\,),\ (x_2,x_3,0\,),\ (x_4,x_5,x_6),$$

respectively. Since $d_{i,j}^2 = g_{i,j}(\xi,\eta)$, we have

$$x_1^2 = 324\,(1-\eta)(1+\eta)$$
$$x_2^2+x_3^2 = 324\,(1-\xi)(1+\xi)$$
$$x_4^2+x_5^2+x_6^2 = (29-18\xi-18\eta)(7+18\xi+18\eta)$$
$$(x_1-x_2)^2+x_3^2 = 36\,(7-3\xi-3\eta)(1+3\xi+3\eta)$$
$$(x_1-x_4)^2+x_5^2+x_6^2 = (17-18\xi)(31+18\xi)$$
$$(x_2-x_4)^2+(x_3-x_5)^2+x_6^2 = (17-18\eta)(31+18\eta).$$

Solve the equation system for $\{x_1, x_2, \cdots, x_6\}$ and receive a manifold solution:

$$x_1 = 18\sqrt{(1-\eta)(1+\eta)},$$
$$x_2 = \frac{11 - 18\xi - 18\eta + 18\xi\eta}{\sqrt{(1-\eta)(1+\eta)}},$$
$$x_3 = \frac{7\sqrt{3}}{\sqrt{(1-\eta)(1+\eta)}},$$
$$x_4 = \frac{18\xi + 11\eta - 18\xi\eta - 18\eta^2}{\sqrt{(1-\eta)(1+\eta)}},$$
$$x_5 = \frac{7\sqrt{3}\,\eta}{\sqrt{(1-\eta)(1+\eta)}},$$
$$x_6 = 7\sqrt{3},$$

where (ξ, η) ranges over

$$G_0 = \{\,(x,y) \in \mathbf{R}^2 \mid G(x,y) = 0,\ |x| < 1,\ |y| < 1\}.$$

Thus, we obtain a family of tetrahedra $T_{(\xi,\eta)}$ with vertices:

$P_1 = (\,0,\,0,\,0\,)$,

$P_2 = (18\sqrt{(1-\eta)(1+\eta)},\,0,\,0\,)$,

$$P_3 = \left(\frac{11 - 18\xi - 18\eta + 18\xi\eta}{\sqrt{(1-\eta)(1+\eta)}},\ \frac{7\sqrt{3}}{\sqrt{(1-\eta)(1+\eta)}},\ 0\right),$$

$$P_4 = \left(\frac{18\xi + 11\eta - 18\xi\eta - 18\eta^2}{\sqrt{(1-\eta)(1+\eta)}},\ \frac{7\sqrt{3}\,\eta}{\sqrt{(1-\eta)(1+\eta)}},\ 7\sqrt{3}\right),$$

which share the same volume, circumradius and face areas,

$$441,\ \frac{43\sqrt{3}}{6},\ 84\sqrt{3},\ 63\sqrt{3},\ 63\sqrt{3},\ 63\sqrt{3},$$

according to Lemma 3. Now, Theorem 1 is proven.

3. CONCLUSION

A negative answer is presented to an open problem proposed in ISSAC 2000 [3]: for *any* given positive real constants V, R, A_1, A_2, A_3, A_4, whether or not there are at most *finitely many* tetrahedra, all having these values as their volume, circumradius and four face areas, respectively. We construct a family of *infinitely many* tetrahedra $T_{(x,y)}$ which all share the same volume, circumradius and four face areas, whenever (x, y) ranges in a one-dimensional manifold. Our example, however, is in the case of $A_2 = A_3 = A_4$.

We conjecture that, for given six positive constants V, R, A_1, A_2, A_3, A_4 where A_1, A_2, A_3, A_4 are pairwise distinct, there are at most nine tetrahedra, all having these values as their volume, circumradius and four face areas, respectively.

What we discussed in this paper can be regarded as a generalized problem on geometric constraint solving that involves not only lengths or angles, but also areas, volumes and circumradius. See [4] for a recent approach to geometric constraint solving with distance geometry.

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One of the referee investigated the conjecture raised in the conclusion using a numerical test at random and always gets nine solutions, albeit some might not have positive values. This constitutes a probabilistic "proof" that the generic finite case has at most nine positive solutions, whenever face areas are pairwise distinct. Furthermore, the referee made another observation. In the generic case of nine solutions, one of them is always observed to give negative values, then a stronger conjecture would be that there are at most eight tetrahedra for a given set of parameter values for which only finitely many exist.

Another referee pointed out that the proofs of Lemma 2 and Theorem 1 could be replaced by alternate ones as follows. Since the set G_0 is compact, Lemma 2 can be proved by showing that none of the six systems of equations $\{G = 0, g_{i,j} = 0\}$ (for $1 \le i < j \le 4$) has a real solution in the range $-\frac{1}{2} \le x, y \le \frac{3}{4}$, and then observing that, there is a point (x_0, y_0) in G_0 such that $g_{i,j}(x_0, y_0) > 0$ for all i, j, $(1 \le i < j \le 4)$. Moreover, the final part of the proof of Theorem 1 could be simplified by appealing to a theorem in Section 40 of Blumenthal's book [1]. That theorem implies that the tetrahedron exists if and only if all the squared volumes $V^2, A_1^2, A_2^2, A_3^2, A_4^2$ have positive values, which therefore is not only necessary but also sufficient condition for existence of the tetrahedron.

5. REFERENCES

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